A General Purpose Technology Explains the Solow Paradox and Wage Inequality: Appendix

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Abstract

This note contains the unpublished appendix of Bas Jacobs and Richard Nahuis (2002), "A General Purpose Technology Explains the Solow Paradox and Wage Inequality", *Economics Letters*, 74, 243-250. The appendix contains derivations of i) the first-order conditions, ii) equilibrium of the model, iii) the conditions for stability, and iv) the derivation of the slopes of the phase lines.

First-order conditions

Firms maximize the discounted value of profits flows $\Pi_j \equiv \int_0^\infty \pi_j \exp[-rt]dt$, subject to the demand function for their variety, $X_j = \left(\frac{p_j}{p_X}\right)^{-\varepsilon} X$, and the technology accumulation constraint given in equation (3) in the text. Instantaneous profits are given by: $\pi_j \equiv p_j X_j - w_L L_j - w_H H_j - rK_j$. Therefore, the current-value Hamiltonian of the optimal control problem for firm j reads as:

$$\mathcal{H}_j = p_j X_j - w_H H_j - w_L L_j - rK_j + q_j B(1-u_j) H_j F_j$$

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Define $\theta \equiv (1-\alpha)\beta$ and $\xi \equiv (1-\alpha)(1-\beta)$. First-order conditions (FOC's) for an optimum are:

$$\frac{\partial \mathcal{H}_j}{\partial L_j} = p_j \frac{\varepsilon - 1}{\varepsilon} \xi A K_j^{\alpha} F_j^{1-\alpha} (u_j H_j)^{\theta} L_j^{\xi - 1} - w_L = 0$$

$$\frac{\partial \mathcal{H}_j}{\partial H_j} = p_j \frac{\varepsilon - 1}{\varepsilon} \theta A K_j^{\alpha} F_j^{1-\alpha} (u_j H_j)^{\theta} H_j^{-1} L_j^{\xi} - w_H + q_j B(1-u_j) F_j = 0$$

$$\frac{\partial \mathcal{H}_j}{\partial u_j} = p_j \frac{\varepsilon - 1}{\varepsilon} \theta A K_j^{\alpha} F_j^{1-\alpha} (u_j H_j)^{\theta} u_j^{-1} L_j^{\xi} - q_j B H_j F_j = 0$$

$$\frac{\partial \mathcal{H}_j}{\partial K_j} = p_j \frac{\varepsilon - 1}{\varepsilon} \alpha A K_j^{\alpha - 1} F_j^{1-\alpha} (u_j H_j)^{\theta} L_j^{\xi} - r = 0$$

$$\frac{\partial \mathcal{H}_j}{\partial F_j} = p_j \frac{\varepsilon - 1}{\varepsilon} (1-\alpha) A K_j^{\alpha} F_j^{-\alpha} (u_j H_j)^{\theta} L_j^{\xi} + q_j B(1-u_j) H_j = r q_j - \dot{q}_j$$

in addition to the transversality condition:

$$\lim_{t \to \infty} F_j \exp\left(-\int_0^t r(v) dv\right) = 0$$

Equilibrium

The second and third FOC's give the no-arbitrage condition for the allocation of time of high-skilled workers in the production of goods and learning in the text.

The differential equation for $R \equiv F/K$ can be obtained using the economy's resource constraint - after imposing symmetric equilibrium:

$$\frac{\dot{R}}{R} = B(1-u)H - spAR^{1-\alpha}(uH)^{\theta} L^{\xi}$$

The derivation of the differential equation describing u requires two additional steps. First, the no-arbitrage condition of high-skilled workers can be differentiated with respect to time to arrive at:

$$\alpha \frac{\dot{R}}{R} + (1-\theta) \frac{\dot{u}}{u} = -\frac{\dot{q}}{q}$$

Second, we can substitute the first term in the last FOC out by rewriting the no-arbitrage condition:

$$(1-\alpha)AR^{-\alpha}(uH)^{\theta}L^{\xi} = \frac{1-\alpha}{\theta}qBuH$$

Using the last three results we obtain the differential equation for u:

$$\frac{\dot{u}}{u} = \frac{1-\alpha}{\theta}BuH + \frac{1-\alpha}{1-\theta}BH + \frac{\alpha(1-sp)}{\theta-1}AR^{1-\alpha}(uH)^{\theta}L^{\xi}$$

Equilibrium follows by setting $\frac{\dot{u}}{u} = 0$ and $\frac{\dot{R}}{R} = 0$ and solving for u^* and R^*.

Stability

The stability of the equilibrium can be checked by evaluating the determinant of Jacobian matrix J at the equilibrium E. The four partial derivatives of J at E are:

$$\left.\frac{\partial \dot{R}}{\partial R}\right|_E = -(1-\alpha)B(1-u^*)H < 0$$

$$\left.\frac{\partial \dot{R}}{\partial u}\right|_E = -BHR^*\left(1+\theta(1-u^*)/u^*\right) < 0$$

$$\begin{aligned}\left.\frac{\partial \dot{u}}{\partial u}\right|_E &= 2\left(\tfrac{1-\alpha}{\theta}\right)BHu^* + \left(\tfrac{1-\alpha}{1-\theta}\right)BH - (1+\theta)\left(\tfrac{(1-\alpha)}{\theta}BHu^* + \tfrac{(1-\alpha)}{(1-\theta)}BH\right) \\ &= \tfrac{1-\alpha}{\theta}BHu^* - \tfrac{\theta\phi}{u^*}\end{aligned}$$

$$\left.\frac{\partial \dot{u}}{\partial R}\right|_E = -(1-sp)\left(\frac{1-\alpha}{1-\theta}\right)\frac{r^*u^*}{R^*} < 0$$

where $\phi \equiv ((1-\alpha)/\theta)BHu^* + ((1-\alpha)/(1-\theta))BH$. The equilibrium is saddle-point stable if $\partial\dot{u}/\partial u > 0$. Then, the determinant of the Jacobian is negative. This will be the case if:

$$\frac{1-\alpha}{\theta}BHu^* - \frac{\theta\phi}{u^*} > 0$$

substitution of ϕ gives:

$$u^* > \left(\frac{\theta}{1-\theta}\right)^2$$

Slopes phase-lines

The slopes of the curves in figure 1 are derived by totally differentiating the $\dot{R} = 0$ and $\dot{u} = 0$ lines with respect to R and u. The $\dot{R} = 0$ locus is downward sloping:

$$\left.\frac{du}{dR}\right|_{\dot{R}=0} = -\frac{u(1-\alpha)/R}{\theta + u/(1-u)} < 0$$

The $\dot{u} = 0$ locus is upward sloping:

$$\left.\frac{du}{dR}\right|_{\dot{u}=0} = \frac{(1-\alpha)\phi/R}{(1-\alpha)BH/\theta - \theta\phi/u} > 0$$

The denominator is positive as a consequence of the stability condition, *i.e.* when $u^* > (\theta/(1-\theta))^2$.