Optimal Income Taxation with Endogenous Human Capital: Appendix

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Household optimization

Consolidating the household budget constraint yields - omitting indices α:

$$c = (1-t)w(1-x) - py + (1-t)wl\phi(.) + 2g.$$

Substitution of the household budget constraint in the utility function yields and unconstrained maximization problem:

$$\max_{\{l,x,y\}} u = \ln\left((1-t)w(1-x) - py + (1-t)wl\phi(.) + 2g - \frac{l^{1+1/\varepsilon}}{1+1/\varepsilon}\right).$$

This is equivalent to maximization of the transformed problem:

$$\max_{\{l,x,y\}} u^* = (1-t)w(1-x) - py + (1-t)wl\phi(.) + 2g - \frac{l^{1+1/\varepsilon}}{1+1/\varepsilon}.$$

First-order conditions are:

$$\frac{\partial u^*}{\partial l} = (1-t)w\phi(.) - l^{1/\varepsilon} = 0,$$
$$\frac{\partial u^*}{\partial x} = (1-t)wl\phi_x(.) - (1-t)w = 0,$$
$$\frac{\partial u^*}{\partial y} = (1-t)wl\phi_y(.) - p = 0.$$

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Rewriting yields:

$$l = [(1-t)w\phi(.)]^{\varepsilon},$$
$$\frac{\phi_x}{\phi_y} = \frac{\gamma y}{\upsilon x} = \frac{(1-t)w}{p},$$
$$l\phi_x = 1.$$

We can solve for the optimal values of l, x, and y. First, use the marginal rate of technical substitution for goods and time invested in education to get:

$$y = \frac{\upsilon w(1-t)}{\gamma p}x.$$

Substitute the last result in the equation for labor supply then we get l as a function of x only:

$$l = [(1-t)w]^{\varepsilon}h^{\varepsilon}x^{\gamma\varepsilon}y^{\upsilon\varepsilon} = [(1-t)w]^{\varepsilon}h^{\varepsilon}x^{\gamma\varepsilon}\left(\frac{\upsilon w(1-t)}{\gamma p}x\right)^{\upsilon\varepsilon}.$$

Simplifying yields:

$$l = \left(\frac{\upsilon}{p\gamma}\right)^{\varepsilon\upsilon} h^{\varepsilon}\left(w(1-t)\right)^{\varepsilon(1+\upsilon)} x^{\varepsilon(\gamma+\upsilon)}.$$

Second, we can rewrite the first-order condition for leisure:

$$l\phi(.) = [(1-t)w]^{\varepsilon}\phi(.)^{1+\varepsilon}.$$

And we have:

$$\gamma l\phi(.) = x,$$

which follows from the arbitrage condition for learning. Substitution of the rewritten first-order condition for leisure in the last expression gives:

$$\gamma[(1-t)w]^{\varepsilon}\phi(.)^{1+\varepsilon} = x,$$

Third, substitute the production function of human capital to obtain:

$$\gamma[(1-t)w]^{\varepsilon}h^{1+\varepsilon}x^{\gamma(1+\varepsilon)}\left(\frac{\upsilon w(1-t)}{\gamma p}x\right)^{\upsilon(1+\varepsilon)} = x.$$

Simplifying results in the expression for the optimal x:

$$x^* = \gamma^{\frac{1}{\mu}}h^{\frac{1+\varepsilon}{\mu}}w^{\frac{\varepsilon+\upsilon(1+\varepsilon)}{\mu}}\left(\frac{\upsilon}{\gamma p}\right)^{\frac{\upsilon(1+\varepsilon)}{\mu}}(1-t)^{\frac{\varepsilon+\upsilon(1+\varepsilon)}{\mu}},$$

where $\mu \equiv 1-(1+\varepsilon)(\gamma+\upsilon) > 0$. y^* and l^* follow from plugging the value for x^* into the equations for y and l.

Second-order conditions

To check the second-order conditions we first derive the utility function as a function of x only. Then, we evaluate the second derivative of the utility function at the optimum. If this second derivative is negative we know that utility reaches a maximum in (x, y, l) space, since optimum values of y and l are positive transformations of x.

Substitution of the optimal values of y and l yields indirect utility as a function of x only. First use:

$$\frac{1}{1+1/\varepsilon} l^{1+1/\varepsilon} = \frac{1}{1+1/\varepsilon} \left([(1-t)w\phi(.)]^{\varepsilon} \right)^{(1+1/\varepsilon)} = \frac{1}{1+1/\varepsilon} [(1-t)w\phi(.)]^{1+\varepsilon}.$$

Second, note that:

$$(1-t)w\phi(.)l = (1-t)w\phi(.) [(1-t)w\phi(.)]^{\varepsilon} = [(1-t)w\phi(.)]^{1+\varepsilon}.$$

Indirect utility is given by:

$$v^* = (1-t)w(1-x) - p\Omega x + (1-t)wl\phi(.) + 2g - \frac{1}{1+1/\varepsilon} l^{1+1/\varepsilon}.$$

Now substitute the expressions for $\frac{1}{1+1/\varepsilon} l^{1+1/\varepsilon}$ and $(1-t)w\phi(.)l$ to obtain:

$$v^* = (1-t)w(1-x) - p\Omega x + \frac{1}{1+\varepsilon} [(1-t)w\Phi(x)]^{1+\varepsilon} + 2g.$$

Where we substituted $y = \frac{\upsilon}{\gamma} \frac{(1-t)w}{p} x \equiv \Omega x$, and $\Phi(x) \equiv \phi(x, \Omega x)$.

The second derivative of the utility function at the optimum values for y and l is:

$$\frac{\partial^2 v^*}{\partial x^2} = ((1-t)w)^{1+\varepsilon} \Phi(x)^{\varepsilon} \left(\varepsilon \frac{\Phi_x^2}{\Phi} + \Phi_{xx} \right).$$

For utility to reach a maximum the term in brackets must be negative, since all other terms are positive. Next use the properties of Φ: $\Phi_x = h(\alpha)(\gamma + \upsilon)\Omega^{\upsilon} x^{\gamma+\upsilon-1}$, and $\Phi_{xx} = h(\alpha)(\gamma+\upsilon)(\gamma+\upsilon-1)\Omega^{\upsilon} x^{\gamma+\upsilon-2}$. Upon substitution in the term in brackets we derive:

$$(1+\varepsilon)(\gamma+\upsilon) < 1.$$

Derivation optimum tax rate

The Lagrangian for maximization of social welfare is given by:

$$\mathcal{L} = \int_{\underline{\alpha}}^{\infty} \left(\Psi(V) + \eta\left(twl\phi(.) + tw(1-x) - G - \Lambda\right)\right) dF(\alpha),$$

where η is the Lagrange multiplier associated with the government budget constraint.

First, we have:

$$\frac{\partial \mathcal{L}}{\partial G} = \int_{\underline{\alpha}}^{\infty} \left(\Psi'\lambda - \eta + \eta t \frac{\partial H}{\partial G}\right) dF = 0.$$

Note that there are no income effects on both labor supply and investment in human capital, so that $\frac{\partial H}{\partial G} = 0$. This equation can be rewritten using the definition of b:

$$\int_{\underline{\alpha}}^{\infty} (b-1)\, dF = 0,$$

so that:

$$\int_{\underline{\alpha}}^{\infty} b dF = \overline{b} = 1.$$

Second, we have:

$$\frac{\partial \mathcal{L}}{\partial t} = \int_{\underline{\alpha}}^{\infty} -\Psi'\left(wl\phi(.) + w(1-x)\right) + \eta\left(wl\phi(.) + w(1-x)\right) + \eta tw\phi(.)\frac{\partial l}{\partial t} + \eta\left(twl\left(\phi_x\frac{\partial x}{\partial t} + \phi_y\frac{\partial y}{\partial t}\right) - tw\frac{\partial x}{\partial t}\right) dF = 0.$$

(If goods invested in education are deductible we would have an additional $-p\frac{\partial y}{\partial t}$ in the last term in brackets). Rewriting yields:

$$\frac{\partial \mathcal{L}}{\partial t} = \int_{\underline{\alpha}}^{\infty} \left(-\frac{\Psi'}{\eta} + 1\right)\left(wl\phi(.) + w(1-x)\right) + tw\phi(.)\frac{\partial l}{\partial t} + twl\left(\phi_x\frac{\partial x}{\partial t} + \phi_y\frac{\partial y}{\partial t}\right) - tw\frac{\partial x}{\partial t} dF = 0.$$

This formula can be simplified in four steps. First, use the definition of ξ to rewrite the first term:

$$\int_{\underline{\alpha}}^{\infty} \left(-\frac{\Psi'}{\eta}+1\right)(wl\phi(.)+w(1-x))\,dF=\xi\int_{\underline{\alpha}}^{\infty} HdF.$$

Second, rewrite the second term:

$$\int_{\underline{\alpha}}^{\infty} tw\phi(.)\frac{\partial l}{\partial t}dF=\frac{t}{1-t}\int_{\underline{\alpha}}^{\infty} wl\phi(.)\frac{1-t}{l}\frac{\partial l}{\partial t}dF=-\frac{t}{1-t}\varepsilon_{lt}\int_{\underline{\alpha}}^{\infty} wl\phi(.)dF.$$

Third, note that from the first order condition for x we have:

$$twl\phi_x\frac{\partial x}{\partial t}-tw\frac{\partial x}{\partial t}=0.$$

And, fourth, rewrite the last term:

$$\int_{\underline{\alpha}}^{\infty} twl\left(\phi_x\frac{\partial x}{\partial t}+\phi_y\frac{\partial y}{\partial t}\right)-tw\frac{\partial x}{\partial t}dF=\int_{\underline{\alpha}}^{\infty} twl\phi_y\frac{\partial y}{\partial t}dF$$
$$=\frac{t}{1-t}\int_{\underline{\alpha}}^{\infty} wl\phi(.)\frac{\phi_y y}{\phi(.)}\frac{1-t}{y}\frac{\partial y}{\partial t}dF=-\frac{t}{1-t}\upsilon\varepsilon_{yt}\int_{\underline{\alpha}}^{\infty} wl\phi(.)dF.$$

(If goods invested in education are deductible this term is zero. In that case the first line would contain the extra term $-tp\frac{\partial y}{\partial t}$. Similarly as above one may then substitute the first order condition for y: $twl\phi_y\frac{\partial y}{\partial t}-tp\frac{\partial y}{\partial t}=0$.)

Substituting all terms in the first-order condition for t gives – after simplifying:

$$\frac{t}{1-t}=\frac{\xi}{\omega(\varepsilon_{lt}+\upsilon\varepsilon_{yt})},$$

where $\omega\equiv\int_{\underline{\alpha}}^{\infty} wl\phi(.)dF/\int_{\underline{\alpha}}^{\infty} wl\phi(.)+w(1-x)dF$ is the average ratio of second period income in total income.