Appendix A

Tensor calculus

A.1 Tensors

The basis for expressing elastic equations is Euclidian 3-dimensional space, i.e., $\mathbb{R}^3$ with the Euclidian inner product. Vectors from $\mathbb{R}^3$ are denoted by bold lower case letters, e.g. $\mathbf{a}$, $\mathbf{b}$, and are also known as first-order tensors, or 1-tensors. We assume that an inner product on $\mathbb{R}^3$ is given, and that it is denoted by $\mathbf{a} \cdot \mathbf{b}$ for $\mathbf{a}$ and $\mathbf{b} \in \mathbb{R}^3$.

If we have a basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ for $\mathbb{R}^3$, then we can determine the components of a vector with regard to that basis using the dual basis. The dual of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is denoted $\{\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3\}$. It is determined uniquely by

$$\mathbf{x} = \sum_i (\mathbf{a}^i \cdot \mathbf{x})\mathbf{a}_i, \qquad \mathbf{x} \in \mathbb{R}^3$$

A basis is called *orthonormal* if it is equal to its own dual.

The set of linear mappings from $\mathbb{R}^3$ to $\mathbb{R}^3$ is a 9-dimensional space, denoted by $\mathrm{Lin}(\mathbb{R}^3, \mathbb{R}^3)$, or Lin for short. The elements of Lin are also known as second order tensors, and are printed in bold upper case, e.g. $\mathbf{T}$ and $\mathbf{E}$. The identity tensor is denoted by $\mathbf{I}$.

When we evaluate a linear mapping $\mathbf{A}$ in a point $\mathbf{x}$, we write $\mathbf{A} \cdot \mathbf{x}$. The product $\mathbf{A} \cdot \mathbf{B}$ of two 2-tensors $\mathbf{A}$ and $\mathbf{B}$ is the 2-tensor defined by

$$(\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{x} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{x}).$$

The transpose (or adjoint) $\mathbf{A}^*$ of $\mathbf{A}$ is the unique tensor satisfying

$$\mathbf{u} \cdot (\mathbf{A} \cdot \mathbf{v}) = (\mathbf{A}^* \cdot \mathbf{u}) \cdot \mathbf{v}. \tag{A.1}$$

For notational convenience, we set $\mathbf{v} \cdot \mathbf{A} := \mathbf{A}^* \cdot \mathbf{v}$. The transpose is a linear operation on 2-tensors. A tensor is called symmetric if $\mathbf{A}^* = \mathbf{A}$.

A function taking $\mathbf{u}$ to $\mathbf{u} \cdot \mathbf{A} \cdot \mathbf{u}$ is called a quadratic form. If $\mathbf{u} \cdot \mathbf{A} \cdot \mathbf{u} > 0$ for all $\mathbf{u} \neq 0$, then the quadratic form is positive definite, and if $\mathbf{u} \cdot \mathbf{A} \cdot \mathbf{u} \geq 0$, then it is positive semidefinite. Negative definite and negative semidefinite are defined similarly.

If $\mathbf{a}$ and $\mathbf{b}$ are 1-tensors, then we can construct a linear mapping from $\mathbf{a}$ and $\mathbf{b}$ by setting

$$(\mathbf{a} \otimes \mathbf{b})(\mathbf{x}) = (\mathbf{b} \cdot \mathbf{x})\mathbf{a}, \qquad \mathbf{a}, \mathbf{b} \in \mathbb{R}^3. \tag{A.2}$$

This mapping is called a dyadic product, *dyad* or tensor-product. To illustrate the meaning, when $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ are orthonormal bases, then the dyad $(\mathbf{a}_2 \otimes \mathbf{b}_1)$ applied to $\mathbf{x}$ takes the magnitude of the $\mathbf{b}_1$ component of $\mathbf{x}$, and maps that component to $\mathbf{a}_2$. If $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ are bases of $\mathbb{R}^3$, then set of dyads given by $\{\mathbf{a}_i \otimes \mathbf{b}_j | i, j = 1, 2, 3\}$ has nine elements, and it forms a basis of the linear mappings of $\mathbb{R}^3$. Since these dyads form a basis, we may also use Equation (A.2) as a definition for function application. Higher order tensor products (3-tensors and 4-tensors) may also defined, to represent mappings between $\mathbb{R}^3$ and Lin and between Lin and Lin.

We can also express matrix multiplication using dyads. Let $\mathbf{A} : \mathbf{a} \otimes \mathbf{b}$ and $\mathbf{B} := \mathbf{c} \otimes \mathbf{d}$, then we have

$$(\mathbf{A} \cdot \mathbf{B})(\mathbf{x}) = (\mathbf{a} \otimes \mathbf{b})\mathbf{c}(\mathbf{d} \cdot \mathbf{x}) = (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \otimes \mathbf{d})(\mathbf{x}).$$

Since the dyads form a base of Lin, we may also define linear operations in Lin in terms of dyads. For instance, the transpose or adjoint can be defined as

$$(\mathbf{a} \otimes \mathbf{b})^* = \mathbf{b} \otimes \mathbf{a}.$$

The trace is a linear functional on Lin: it takes a linear mapping, and returns a number. It can be defined in terms of dyads

$$\mathrm{trace}(\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}. \tag{A.3}$$

With the help of the trace operator we can define an inner product on Lin. The inner product between $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} : \mathbf{B}$, and is given by

$$\mathbf{A} : \mathbf{B} = \mathrm{trace}(\mathbf{B}^* \cdot \mathbf{A}). \tag{A.4}$$

We have $\mathrm{trace}(\mathbf{A}) = \mathbf{A} : \mathbf{I}$. When 2-tensors are represented by matrices, then the trace corresponds to the sum of the diagonal elements. The inner product on Lin corresponds to the Euclidian inner product on $\mathbb{R}^{3\times 3}$.

By use of tensor products, we may extend these inner products to tensor products between arbitrary dimensions. The inner product of a k and l-tensor product

$$(\mathbf{a}_1 \otimes \cdots \otimes \mathbf{a}_k) \cdot (\mathbf{b}_1 \otimes \cdots \otimes \mathbf{b}_l) = (\mathbf{a}_k \cdot \mathbf{b}_1)(\mathbf{a}_1 \otimes \cdots \otimes \mathbf{a}_{k-1}) \otimes (\mathbf{b}_2 \otimes \cdots \otimes \mathbf{b}_l).$$

Similarly, we may extend the inner product for two tensors to arbitrary dimensions.

$$(\mathbf{a}_1 \otimes \cdots \otimes \mathbf{a}_k) : (\mathbf{b}_1 \otimes \cdots \otimes \mathbf{b}_l) = (\mathbf{a}_{k-1} \cdot \mathbf{b}_1)(\mathbf{a}_k \cdot \mathbf{b}_2)(\mathbf{a}_1 \otimes \cdots \otimes \mathbf{a}_{k-2}) \otimes (\mathbf{b}_3 \otimes \cdots \otimes \mathbf{b}_l).$$

The expression

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

measures the oriented volume of the parallelepiped spanned by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. It is called the *scalar triple-product* of vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. The determinant of a mapping $\mathbf{A}$ from

Lin measures how the volume of a parallelepiped changes when it transformed through $\mathbf{A}$. This definition is independent of the parallelepiped used. In other words, given an arbitrary set of independent vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, then the determinant is uniquely defined by

$$\det \mathbf{A} = \frac{(\mathbf{A}\mathbf{a}) \cdot (\mathbf{A}\mathbf{b} \times (\mathbf{A}\mathbf{c}))}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}. \tag{A.5}$$

The determinant satisfies $\det(\mathbf{A} \cdot \mathbf{B}) = \det \mathbf{A} \det \mathbf{B}$, and hence $\det(\mathbf{A}^{-1}) = 1/\det \mathbf{A}$, if $\mathbf{A}^{-1}$ exists. It follows that $\det(\mathbf{X}\mathbf{A}\mathbf{X}^{-1}) = \det(\mathbf{A})$: the determinant is invariant under a change of basis.

A vector $\mathbf{v}$ is called an eigenvector of $\mathbf{A}$ if there is a number λ, the eigenvalue, such that

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{c}.$$

Eigenvalues are given by the roots of the *characteristic polynomial.* The characteristic polynomial of a 2-tensor $\mathbf{A}$ is defined as $\det(\mathbf{A} - \lambda\mathbf{I})$. We can expand this expression as a polynomial, thus obtaining

$$\det(\mathbf{A} - \lambda\mathbf{I}) = -\lambda^3 + \iota_1\lambda^2 - \iota_2\lambda + \iota_3.$$

The coefficients ι_1, ι_2 and ι_3 in this expansion are called *invariants* of $\mathbf{A}$. Since the determinant is invariant under change of basis, the invariants also are. If λ_1, λ_2 and λ_3 are the eigenvalues of $\mathbf{A}$, then we have

$$\begin{aligned} \iota_1(\mathbf{A}) &= \lambda_1 + \lambda_2 + \lambda_3, \\ \iota_2(\mathbf{A}) &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \\ \iota_3(\mathbf{A}) &= \lambda_1\lambda_2\lambda_3. \end{aligned} \tag{A.6}$$

The invariants can also be determined directly from $\mathbf{A}$. We have

$$\begin{aligned} \iota_1(\mathbf{A}) &= \operatorname{trace}(\mathbf{A}), \\ \iota_2(\mathbf{A}) &= \frac{1}{2}((\operatorname{trace} \mathbf{A})^2 - \operatorname{trace}(\mathbf{A}^* \cdot \mathbf{A})), \\ \iota_3(\mathbf{A}) &= \det \mathbf{A}. \end{aligned}$$

A.2 Tensor calculus

A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable in $x \in \mathbb{R}$, when there is a number $d(x)$ and a function $r(x, h)$, such that

$$f(x + h) = f(x) + d(x)h + r(x, h), \qquad r(x, h) = o(h) \text{ when } h \to 0.$$

In other words, f is differentiable in x if it may be linearly approximated in a neighborhood of x. The function $d(x)$ is the derivative of f in x, also denoted by $\frac{df}{dx}$.

This definition can be generalized to higher dimensions. Let $\mathcal{V}$ and $\mathcal{W}$ be finite-dimensional vector spaces. A function $F : \mathcal{V} \to \mathcal{W}$, is called Fréchet-differentiable in v if there is a linear mapping $L : \mathcal{V} \to \mathcal{W}$ such that

$$F(v + h) = F(v) + L(h) + o(h), \quad h \in \mathcal{V}.$$

Since L is a linear mapping, we may write it as some product of some D in the tensor product space of $\mathcal{V}$ and $\mathcal{W}$. This representant D of the mapping L is called the derivative.

For example, if a function f maps vectors from $\mathbb{R}^3$ to numbers, then it is differentiable in $\mathbf{x}$ if there exists functions $f_{\mathbf{x}}$ and r, such that

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + f_{\mathbf{x}}(\mathbf{x}) \cdot \mathbf{h} + r(\mathbf{x}, \mathbf{h}),$$

and $r(\mathbf{x}, \mathbf{h}) = o(\|\mathbf{h}\|)$. The mapping $\mathbf{h} \mapsto f_{\mathbf{x}}(\mathbf{x}) \cdot \mathbf{h}$ is a linear mapping. The function can be represented as an inner product of $\mathbf{f}_{\mathbf{x}}(\mathbf{x})$ and the argument. Hence, $\mathbf{f}_{\mathbf{x}}(\mathbf{x})$ is is called the derivative or gradient of f. It is denoted as $\frac{\partial f}{\partial \mathbf{x}}$. Other notations include grad f or ∇f.

If $f : \text{Lin} \to \mathbb{R}$ is a differentiable function taking linear mappings to scalars, then there exists a function r and $\partial f / \partial \mathbf{A} : \text{Lin} \to \mathbb{R}$, such that

$$f(\mathbf{A}+\mathbf{H}) = f(\mathbf{A}) + \frac{\partial f}{\partial \mathbf{A}} : \mathbf{H} + r(\mathbf{A}, \mathbf{H}), \quad \mathbf{A} \in \text{Lin},$$

where $r(\mathbf{A}, \mathbf{H})/\|\mathbf{H}\| \to 0$ when $\mathbf{H} \to 0$. The derivative $\frac{\partial f(\mathbf{A})}{\partial \mathbf{A}}$ is also denoted by $f_{\mathbf{A}}(\mathbf{A})$.

Some derivatives of standard functions are given here.

$$\frac{\partial\, \text{trace}(\mathbf{C})}{\partial \mathbf{C}} = \mathbf{I},$$
$$\frac{\partial \det(\mathbf{C})}{\partial \mathbf{C}} = \det \mathbf{C}\mathbf{C}^{-*},$$
$$\frac{\partial \iota_2(\mathbf{C})}{\partial \mathbf{C}} = \text{trace}(\mathbf{C})\mathbf{I} - \mathbf{C},$$

The inverse of a 2-tensor is another 2-tensor, so taking inverses is a function from Lin to Lin. Its derivative is a linear function from Lin to Lin, which may be represented as a 4-tensor. To avoid the hassle of representing 4-tensors, we simply give the derivative applied to some $\mathbf{H} \in \text{Lin}$:

$$\frac{\partial \mathbf{C}^{-1}}{\partial \mathbf{C}} : \mathbf{H} = -\mathbf{C}^{-1} \cdot \mathbf{H} \cdot \mathbf{C}^{-1}.$$

We mention one differential operator that we shall encounter further, the divergence. The divergence of a vector field $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$\text{div}\, \mathbf{f} = \text{trace}(\partial \mathbf{f}/\partial \mathbf{x}).$$

The divergence of a tensor field $\mathbf{T}$ is defined by

$$\text{div}\, \mathbf{T} = \partial \mathbf{T}/\partial \mathbf{x} : \mathbf{I}.$$

In one dimension, the value of an integral over an interval of a continuous function is given by the values of its primitive at the boundaries of that interval. A similar theorem holds in higher dimensions. If Ψ is a tensor valued function, and continuously differentiable on its domain Ω, and continuous on the closure of Ω, then

$$\int_{\Omega} \partial\Psi/\partial\boldsymbol{x}\, dv(\boldsymbol{x}) = \int_{\partial\Omega} \Psi \otimes \boldsymbol{n}(\boldsymbol{x})\, da(\boldsymbol{x}),$$

where $\boldsymbol{n}$ is the outward pointing normal on $\partial\Omega$. This theorem can applied to tensors of different order, e.g. the divergence theorem. One form that we employ is

$$\int_{\mathcal{B}} \operatorname{div} \mathbf{T}\, dv(\boldsymbol{x}) = \int_{\partial\mathcal{B}} \mathbf{T} \cdot \boldsymbol{n}\, dv(\boldsymbol{x}). \tag{A.7}$$